

On finite groups whose Sylow subgroups are submodular

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Abstract

A subgroup H of a finite group G is called submodular in G , if we can connect H with G by a chain of subgroups, each of which is modular (in the sense of Kurosh) in the next. If a group G is supersoluble and every Sylow subgroup of G is submodular in G , then G is called strongly supersoluble. The properties of groups with submodular Sylow subgroups are obtained. In particular, we proved that in a group every Sylow subgroup is submodular if and only if the group is Ore dispersive and every its biprimary subgroup is strongly supersoluble.

Keywords: finite group, modular subgroup, submodular subgroup, strongly supersoluble group, Ore dispersive group.

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Introduction

Throughout this paper, all groups are finite. The notion of a normal subgroup takes a central place in the theory of groups. One of its generalizations is the notion of a modular subgroup, i.e. a modular element (in the sense of Kurosh [1, Chapter 2, p. 43]) of a lattice of all subgroups of a group. Recall that a subgroup M of a group G is called modular in G , if the following hold:

- 1) $\langle X, M \cap Z \rangle = \langle X, M \rangle \cap Z$ for all $X \leq G, Z \leq G$ such that $X \leq Z$, and
- 2) $\langle M, Y \cap Z \rangle = \langle M, Y \rangle \cap Z$ for all $Y \leq G, Z \leq G$ such that $M \leq Z$.

Properties of modular subgroups were studied in the book [1]. Groups with all subgroups are modular were studied by R. Schmidt [1], [2] and I. Zimmermann [3]. By parity of reasoning with subnormal subgroup, in [3] the notion of a submodular subgroup was introduced.

Definition [3]. A subgroup H of a group G is called submodular in G , if there exists a chain of subgroups $H = H_0 \leq H_1 \leq \dots \leq H_{s-1} \leq H_s = G$ such that H_{i-1} is a modular subgroup in H_i for $i = 1, \dots, s$.

If $H \neq G$, then the chain can be compacted to maximal modular subgroups.

It's well known that in a nilpotent group every Sylow subgroup is normal (subnormal). In the paper [3] groups with submodular subgroups were studied. In particular, it was proved that in a supersoluble group G every Sylow subgroup is submodular if and only if $G/F(G)$ is abelian of squarefree exponent. A criterion of the submodularity of Sylow subgroups in an arbitrary group was found.

This paper is devoted to the further study of groups with submodular Sylow subgroups.

A group we will call strongly supersoluble, if it is supersoluble and every its Sylow subgroup is submodular in it.

The class of all strongly supersoluble groups we will denote $s\mathfrak{U}$. We proved that the class of groups $s\mathfrak{U}$ is a hereditary local formation. We obtained that a group is strongly supersoluble if and only if it is metanilpotent and every its Sylow subgroup is submodular. The class of all groups with submodular Sylow subgroups we denote $sm\mathfrak{U}$. We proved that $sm\mathfrak{U}$ forms a hereditary local formation and its local screen was found. We established that in a group every Sylow subgroup is submodular if and only if the group is Ore dispersive and every its biprimary subgroup is strongly supersoluble.

1. Preliminaries

We use the standard notation and terminology (see [4] and [5]). Recall some of them.

Let G be a group. $Syl_p(G)$ is a set of all Sylow p -subgroups of G for some prime p ; $Syl(G)$ is a set of all Sylow subgroups of G ; M_G is the core of subgroup M of G , i.e. the intersection of all subgroups conjugated with M in G ; $F(G)$ is the Fitting subgroup of G , i.e. the product of all normal nilpotent subgroups of G ; $F_p(G)$ is a p -nilpotent radical of G , i.e. the product of all normal p -nilpotent subgroups of G , p is some prime.

A group G of order $p_1^{n_1}p_2^{n_2}\dots p_n^{n_k}$ is called Ore dispersive [4, p. 251], if $p_1 > p_2 > \dots > p_n$ and G has a normal subgroup of order $p_1^{n_1}p_2^{n_2}\dots p_n^{n_i}$ for every $i = 1, 2, \dots, k$.

We use the following notation for concrete classes of group: \mathfrak{S} is the class of all soluble groups; \mathfrak{U} is the class of all supersoluble groups; \mathfrak{N} is the class of all nilpotent groups; $\mathfrak{A}(p-1)$ is the class of all abelian groups of exponent dividing $p-1$. By $\mathcal{M}(\mathfrak{X})$ is denoted the class of all minimal non \mathfrak{X} -groups, i.e. such groups G for which all proper subgroups of G are contained in \mathfrak{X} , but $G \notin \mathfrak{X}$.

A class of groups \mathfrak{F} is called a *formation* if the following conditions hold: (a) every quotient group of a group lying in \mathfrak{F} also lies in \mathfrak{F} ; (b) if $H/A \in \mathfrak{F}$ and $H/B \in \mathfrak{F}$ then $H/A \cap B \in \mathfrak{F}$.

A formation \mathfrak{F} is called *hereditary* whenever \mathfrak{F} together with every group contains all its subgroups, and *saturated*, if $G/\Phi(G) \in \mathfrak{F}$ implies that $G \in \mathfrak{F}$.

The \mathfrak{F} -residual of a group G for nonempty formation is denoted by $G^{\mathfrak{F}}$, i.e. the smallest normal subgroup of G with $G/G^{\mathfrak{F}} \in \mathfrak{F}$.

A function $f : \mathbb{P} \rightarrow \{\text{formations}\}$ is called a *local screen*. A formation \mathfrak{F} is called *local*, if there exists a local screen f such that \mathfrak{F} coincides with the class of groups $(G|G/C_G(H/K)) \in f(p)$ for every chief factor H/K of G and $p \in \pi(H/K)$. It's denoted by $\mathfrak{F} = LF(f)$.

Recall that a subgroup H of G is called maximal modular in G , if H is modular in G and from $H \leq M < G$ it always follows $H = M$ for every modular subgroup M in G .

Lemma 1.1 [3, Lemma 1]. *Let G be a group and $T \leq G$. Then the following hold:*

- 1) if T is submodular in G and $U \leq G$, then $U \cap T$ is submodular in U ;
- 2) if T is submodular in G , N is normal in G and $N \leq T$, then T/N is submodular in G/N ;
- 3) if T/N is submodular in G/N , then T is submodular in G ;
- 4) if T is submodular in G , then T^x is submodular in G for every $x \in G$;
- 5) if T_1 and T_2 are submodular subgroups in G , then $T_1 \cap T_2$ is a submodular subgroup in G ;

6) if T is submodular in G , then TN is submodular in G for every normal in G subgroup N .

Lemma 1.2 [2, Lemma 1]. A subgroup M of a group G is maximal modular in G if and only if either M is maximal normal subgroup in G , or G/M_G is nonabelian of order pq , where p and q are primes.

Lemma 1.3 [6, Lemma 2]. Let $G = AB$ be a product of nilpotent subgroups A and B , and G has a minimal normal subgroup N such that $N = C_G(N) \neq G$. Then

- 1) $A \cap B = 1$;
- 2) $N \subseteq A \cup B$;
- 3) if $N \leq A$, then A is a p -group for some prime p and B is p' -group.

Lemma 1.4 [4, Lemma 3.9 (1)]. If H/K is a chief factor of a group G and $p \in \pi(H/K)$, then $G/C_G(H/K)$ is not contain nonidentity normal p -subgroups, and besides $F_p(G) \leq C_G(H/K)$.

Theorem 1.5 [7, Theorem 1.4]. Let H/K be a p -chief factor of a group G . Then $|H/K| = p$ if and only if $\text{Aut}_G(H/K)$ is abelian of exponent dividing $p - 1$.

Theorem 1.6 [5, Chapter IV, Theorem 4.6]. A formation is saturated if and only if it is local.

Lemma 1.7 [4, Lemma 4.5]. Let f be a local screen of a formation \mathfrak{F} . A group G belongs to \mathfrak{F} if and only if $G/F_p(G) \in f(p)$ for every $p \in \pi(G)$.

We need the following property of the class of all supersoluble groups (see, for example, [4, p. 35] or [5, p. 358]).

Lemma 1.8. The class of all supersoluble groups has a local screen f such that $f(p) = \mathfrak{A}(p - 1)$ for every prime p .

2. Strongly supersoluble groups

Lemma 2.1. Let p be the largest prime divisor of $|G|$ and $G_p \in \text{Syl}_p(G)$. If G_p is submodular subgroup in G , then G_p is normal in G .

Proof. We will use an induction by $|G|$. We can consider that $G_p \neq G$ and there exists a chain of subgroups $G_p = H_0 < H_1 < \dots < H_{s-1} < H_s = G$ such that H_{i-1} is maximal modular subgroup in H_i for $i = 1, \dots, s$. By induction, G_p is normal in $H_{s-1} = M$. By Lemma 1.2, either M is normal in G , or G/M_G is nonabelian of order rq , where r and q are primes. In the first case, G_p is normal in G . So let $|G/M_G| = rq$ and G/M_G be a nonabelian group. It follows $|G : M|$ is a prime different from p . So we can assume that $|G : M| = q \neq p$. If $N_G(G_p) \neq G$, then by the Theorem of Sylow $|G : M| = |G : N_G(G_p)| = q \equiv 1 \pmod{p}$. We got a contradiction with $q < p$. So $N_G(G_p) = G$. Lemma is proved.

Corollary 2.1.1 [3, Proposition 9]. If every Sylow subgroup of the group G is submodular in G , then G is Ore dispersive.

Definition 2.2. A group G we will call strongly supersoluble if G is supersoluble and every Sylow subgroup of G is submodular in G .

Denote $s\mathfrak{U}$ the class of all strongly supersoluble groups.

Proposition 2.3 [3, Proposition 10]. A group G is strongly supersoluble if and only if G is supersoluble and $G/F(G)$ is abelian of squarefree exponent.

In the paper we denote \mathfrak{B} the class of all abelian groups of exponent free from squares of primes.

Lemma 2.4. *The class of groups \mathfrak{B} is a hereditary formation.*

Proof. It's clear, if $G \in \mathfrak{B}$, then $H \in \mathfrak{B}$ and $G/N \in \mathfrak{B}$ for any subgroup H and any normal subgroup N of G .

Let G be a group of the smallest order such that $G/N_i \in \mathfrak{B}, N_i \trianglelefteq G, i = 1, 2$, but $G/N_1 \cap N_2 \notin \mathfrak{B}$.

If $N_1 \cap N_2 \neq 1$, then in $N_1 \cap N_2$ there is a normal subgroup K in G . From $|G/K| < |G|$ and $G/K/N_i/K \simeq G/N_i \in \mathfrak{B}, i = 1, 2$, it follows $G/K/(N_1/K \cap N_2/K) \simeq G/N_1 \cap N_2 \in \mathfrak{B}$. This contradicts the choice of G .

Let $N_1 \cap N_2 = 1$. Since \mathfrak{A} is a formation and $G/N_i \in \mathfrak{B} \subseteq \mathfrak{A}$, then $G \in \mathfrak{A}$. Let's show that the exponent of G is free from squares of primes. Let z be an element of order q^n from G , where q is a prime, and $Z = \langle z \rangle$. Assume that $Z \cap N_1 \neq 1$ and $Z \cap N_2 \neq 1$. Since Z is cyclic q -group, then there exists $i \in \{1, 2\}$ such that $Z \cap N_i \leq Z \cap N_{3-i}$. Hence we get a contradiction $1 \neq Z \cap N_i \leq (Z \cap N_1) \cap (Z \cap N_2) = 1$. Hence $Z \cap N_j = 1$ for some $j \in \{1, 2\}$. Then from $G/N_j \in \mathfrak{B}$ and $ZN_j/N_j \simeq Z$ it follows $n < 2$ and $G \in \mathfrak{B}$. Lemma is proved.

Note that the class of groups \mathfrak{B} is not saturated. For example, a cyclic group $G = \langle z | z^4 = 1 \rangle \notin \mathfrak{B}$, but $G/\Phi(G) \in \mathfrak{B}$.

Theorem 2.5. *Let G be a group. Then the following hold:*

- 1) if G is strongly supersoluble, then every subgroup of G is strongly supersoluble;
- 2) if G is strongly supersoluble and $N \trianglelefteq G$, then G/N is strongly supersoluble;
- 3) if $N_i \trianglelefteq G$ and G/N_i is strongly supersoluble for $i = 1, 2$, then $G/N_1 \cap N_2$ is strongly supersoluble;
- 4) if $H_i \trianglelefteq G$, H_i is strongly supersoluble, $i = 1, 2$ and $H_1 \cap H_2 = 1$, then $H_1 \times H_2$ is strongly supersoluble;
- 5) if $G/\Phi(G)$ is strongly supersoluble, then G is strongly supersoluble;
- 6) the class of groups $s\mathfrak{U}$ is a hereditary saturated formation.

Proof. Show the validity of 1). Let $G \in s\mathfrak{U}$ and $H \leq G$. In view of Proposition 2.3 and Lemma 2.4, from $G/F(G) \in \mathfrak{B}$ it follows that $H/H \cap F(G) \simeq HF(G)/F(G) \in \mathfrak{B}$. Since \mathfrak{B} is a homomorph and $H \cap F(G) \leq F(H)$, we get $H/F(H) \simeq H/H \cap F(G)/F(H) \in \mathfrak{B}$. By Proposition 2.3, $H \in s\mathfrak{U}$.

Prove Statement 2). Let $G \in s\mathfrak{U}$ and $N \trianglelefteq G$. By Proposition 2.3, $G/F(G) \in \mathfrak{B}$. Since \mathfrak{B} is a homomorph, from $F(G)N/N \leq F(G/N) = F/N$ we conclude $G/N/F(G/N) \simeq G/F \simeq G/F(G)/F/F(G) \in \mathfrak{B}$. By Proposition 2.3, $G/N \in s\mathfrak{U}$.

Prove Statement 3). Let G be a group of the smallest order such that $N_i \trianglelefteq G$ and $G/N_i \in s\mathfrak{U}$ for $i = 1, 2$, but $G/N_1 \cap N_2 \notin s\mathfrak{U}$. Since \mathfrak{U} is a formation, $G/N_1 \cap N_2 \in \mathfrak{U}$.

If $N_1 \cap N_2 \neq 1$, then take from $N_1 \cap N_2$ a subgroup $K \trianglelefteq G$. From the choice of G and $G/K/N_i/K \simeq G/N_i \in s\mathfrak{U}$ for $i = 1, 2$ it follows $G/K/(N_1/K \cap N_2/K) \simeq G/N_1 \cap N_2 \in s\mathfrak{U}$. This contradicts the choice of G .

Let $N_1 \cap N_2 = 1$. For every Sylow p -subgroup P of G a quotient group $PN_i/N_i \in \text{Syl}_p(G/N_i), i = 1, 2$. From the strongly supersolubility of G/N_i it follows PN_i/N_i is submodular in $G/N_i, i = 1, 2$. By 3) of Lemma 1.1, PN_i is submodular in $G, i = 1, 2$. From properties of Sylow subgroups and 5) of Lemma 1.1, it follows $PN_1 \cap PN_2 = P(N_1 \cap N_2) = P$ is submodular in G . So $G \in s\mathfrak{U}$. This contradicts the choice of G . Statement 3) is proved.

Statement 4) follows from 3).

Prove Statement 5). Let $G/\Phi(G) \in s\mathfrak{U}$. From $s\mathfrak{U} \subseteq \mathfrak{U}$ and the saturation of the class of groups \mathfrak{U} it follows $G \in \mathfrak{U}$. So $F(G/\Phi(G)) = F(G)/\Phi(G)$. Then $G/F(G) \simeq G/\Phi(G)/F(G/\Phi(G)) \in \mathfrak{B}$, i.e. $G \in s\mathfrak{U}$.

Statement 6) follows from 1)–3) and 5). Theorem is proved.

Theorem 2.6. *The class of all strongly supersolubility groups is a local formation and has a local screen f such that $f(p) = \mathfrak{A}(p-1) \cap \mathfrak{B}$ for any prime p .*

Proof. Since $f(p) = \mathfrak{A}(p-1) \cap \mathfrak{B}$ is a formation, f is a local screen. Let a local formation $LF(f)$ be defined by a screen f . Let's show that $s\mathfrak{U} = LF(f)$.

Let $G \in s\mathfrak{U}$ and H/K be any chief factor of G . From $G/F(G) \in \mathfrak{B}$ and $F(G) \leq C_G(H/K)$ it follows $G/C_G(H/K) \in \mathfrak{B}$. Since G is supersoluble, $|H/K| = p$ is some prime. By Lemma 1.5, $G/C_G(H/K) \in \mathfrak{A}(p-1)$. So $G/C_G(H/K) \in f(p)$. Then $G \in LF(f)$ and $s\mathfrak{U} \subseteq LF(f)$.

Let now $G \in LF(f)$. Then $G/C_G(H/K) \in f(p) \subseteq \mathfrak{B}$ for any chief factor H/K of G and $p \in \pi(H/K)$. Since \mathfrak{B} is a formation, we conclude $G/F(G) \in \mathfrak{B}$. So $LF(f) \subseteq s\mathfrak{U}$. Thus $LF(f) = s\mathfrak{U}$. Theorem is proved.

Theorem 2.7. *Let the group $G = AB$ be the product of nilpotent subgroups A and B . If A and B are submodular in G , then G is strongly supersoluble.*

Proof. Let G be a counterexample of minimal order to Theorem. Then, by Theorem of Wielandt-Kegel [8, 9], G is soluble. Let N be a minimal normal subgroup of G . Then $AN/N \simeq A/A \cap N \in \mathfrak{N}$, $BN/N \simeq B/B \cap N \in \mathfrak{N}$. By 6) and 2) of Lemma 1.1, AN/N and BN/N are submodular in G/N . By the choice of G , it follows $G/N \in s\mathfrak{U}$. By 3)–5) of Theorem 2.5, we conclude that N is the only minimal normal subgroup in G and $\Phi(G) = 1$. Then $G = MN$, where M is a maximal subgroup in G , $M \cap N = 1$, $N = C_G(N)$ and $|N| = p^n$ for some prime p . By 1) of Lemma 1.3, $A \cap B = 1$. From $N \subseteq A \cup B$ it follows either $N \leq A$, or $N \leq B$. Without loss of generality, we may suppose $N \leq A$. Then, by 3) of Lemma 1.3, A is a p -subgroup, B is a p' -subgroup.

Let q be the largest prime divisor of $|G|$. If $q \neq p$, then B has some Sylow q -subgroup Q of G . From $Q \trianglelefteq B$ and the submodularity of B in G it follows that Q is submodular in G . By Lemma 2.1, $Q \trianglelefteq G$. Then $N \leq Q$. We get a contradiction with $q \neq p$. So $q = p$. In view of Lemma 2.1, $A \trianglelefteq G$. By Lemma 1.4, $O_p(M) = 1$. Then $M \cap A = 1$ and $A = N$, B is a maximal subgroup of G and $B_G = 1$. Hence, B is a maximal modular subgroup in G . In view of $B_G = 1$ by Lemma 1.2, we conclude $|G| = pr$, where r is a prime and $p \neq r$. So $G \in s\mathfrak{U}$. This contradicts the choice of G . Theorem is proved.

In Theorem 2.7 we can't discard the submodularity of one of subgroups.

Example 2.8. In group $G = AB$, where $A \simeq Z_{17}$ and $B \simeq \text{Aut}(Z_{17}) \simeq Z_{16}$, the subgroup A is submodular, but the subgroup B is not submodular in G . The group G is supersoluble, but not strongly supersoluble. The example also shows that $s\mathfrak{U} \neq \mathfrak{U}$.

Theorem 2.9. *A group G is strongly supersoluble if and only if G is metanilpotent and any Sylow subgroup of G is submodular in G .*

Proof. Necessity follows from that the strongly supersoluble group is supersoluble, and so it has a nilpotent commutator subgroup, i.e. it is metanilpotent.

Sufficiency. Let there exists metanilpotent groups which have all Sylow subgroups are submodular in a group, but the group is not strongly supersoluble. Let's choose from them a group G of the smallest order. Let N be a minimal normal subgroup of G . Then

$G/N \in s\mathfrak{U}$ in view of the choice of G . Since, by Theorem 2.5, the class $s\mathfrak{U}$ is a saturated formation, then N is the only minimal normal subgroup in G and $\Phi(G) = 1$. So $N = C_G(N)$ and $G = NM$, where M is a maximal subgroups of G , $M \cap N = 1$. From the metanilpotency of G and $N = F(G)$ it follows $G/N \simeq M \in \mathfrak{N}$. Let p be the largest prime divisor of $|G|$. Since G is Ore dispersive, it follows that N is contained in some Sylow p -subgroup of G . In view of $O_p(M) = 1$, we conclude that $N \in \text{Syl}_p(G)$ and M is a p' -group. Let $S \in \text{Syl}_q(M)$.

If $G = SN$, then, by Theorem 2.7, G is strongly supersoluble. This contradicts the choice of G .

Let $G \neq SN$ for every $S \in \text{Syl}_q(M)$. Denote $L = SN$. Then L is strongly supersoluble by the choice of G . From $C_G(N) = N$ it follows $O_{p'}(L) = 1$. Then $N = F_p(L)$. By Lemma 1.7 and Theorem 2.6, we get $S \simeq L/F_p(L) \in \mathfrak{A}(p-1) \cap \mathfrak{B}$. Hence and from the nilpotency of M it follows that $M \in \mathfrak{A}(p-1)$. Since $N = F_p(G)$, we conclude that $M \simeq G/F_p(G) \in \mathfrak{A}(p-1)$. By Lemmas 1.7 and 1.8, G is supersoluble. By Definition 2.2, G is strongly supersoluble. This contradicts the choice of G . Theorem is proved.

3. Groups with submodular Sylow subgroups

Denote $sm\mathfrak{U} = (\text{ } G \mid \text{every Sylow subgroup of the group } G \text{ is submodular in } G \text{ })$.

Theorem 3.1. *Let G be a group. Then the following hold:*

- 1) if $G \in sm\mathfrak{U}$ and $H \leq G$, then $H \in sm\mathfrak{U}$;
- 2) if $G \in sm\mathfrak{U}$ and $N \trianglelefteq G$, then $G/N \in sm\mathfrak{U}$;
- 3) if $N_i \trianglelefteq G$ and $G/N_i \in sm\mathfrak{U}$, $i = 1, 2$, then $G/N_1 \cap N_2 \in sm\mathfrak{U}$;
- 4) if $H_i \in sm\mathfrak{U}$, $H_i \trianglelefteq G$, $i = 1, 2$ and $H_1 \cap H_2 = 1$, then $H_1 \times H_2 \in sm\mathfrak{U}$;
- 5) if $G/\Phi(G) \in sm\mathfrak{U}$, then $G \in sm\mathfrak{U}$;
- 6) the class of groups $sm\mathfrak{U}$ is a hereditary saturated formation.

Proof. The validity of Statements 1) and 2) of Theorem follows from Statements 1), 2) and 6) of Lemma 1.1, in view of $H \cap G_p \in \text{Syl}_p(H)$ for some $G_p \in \text{Syl}_p(G)$ and $R/N = G_q N/N \in \text{Syl}_q(G/N)$ for some $G_q \in \text{Syl}_q(G)$.

Prove Statement 3). Let G be a group of the smallest order such that $G/N_i \in sm\mathfrak{U}$, $N_i \trianglelefteq G$, $i = 1, 2$, but $G/N_1 \cap N_2 \notin sm\mathfrak{U}$.

We can suppose that $N_1 \cap N_2 = 1$. Let $P \in \text{Syl}_p(G)$. Then $PN_i/N_i \in \text{Syl}_p(G/N_i)$, $i = 1, 2$. So PN_i/N is submodular in G/N_i . By 3) of Lemma 1.1, PN_i is submodular in G . By the property of Sylow subgroups and Statement 5) of Lemma 1.1, $PN_1 \cap PN_2 = P(N_1 \cap N_2) = P$ is submodular in G , i.e. $G \in sm\mathfrak{U}$. This contradiction completes the proof of 3).

Statement 4) follows from 3).

Prove Statement 5). Let G be a group of the smallest order such that $G/\Phi(G) \in sm\mathfrak{U}$, but $G \notin sm\mathfrak{U}$. Then G is soluble in view of Corollary 2.1.1 and the solubility of $\Phi(G)$. Let N be a minimal normal subgroups of G . From $\Phi(G)N/N \subseteq \Phi(G/N)$ and by Statement 2) of Theorem, we conclude that $G/N/\Phi(G/N) \in sm\mathfrak{U}$. Since $|G/N| < |G|$, $G/N \in sm\mathfrak{U}$. From Statement 3) it follows that N is the only minimal subgroup of G , $|N| = p^n$ for some prime p and $O_{p'}(G) = 1$. Hence, $N \subseteq \Phi(G)$.

Let $Q \in \text{Syl}_q(G)$. From $QN/N \in \text{Syl}_q(G/N)$ it follows that QN/N is submodular in G/N . By Statement 2) of Lemma 1.1, QN is submodular in G .

If $p = q$, then $QN = Q$ is submodular in G .

Let $p \neq q$. Let's consider 2 cases:

(1) $|\pi(G)| = 2$. Then $G/N = QN/N \cdot P/N$, where $P \in \text{Syl}_p(G)$. By Theorem 2.7, G/N is strongly supersoluble. Since the class $s\mathfrak{U}$ of all strongly supersoluble groups is a saturated formation by Theorem 2.5, then from $G/\Phi(G) \simeq G/N/\Phi(G)/N \in s\mathfrak{U}$ it follows that $G \in s\mathfrak{U} \subseteq sm\mathfrak{U}$. This contradicts the choice of G .

(2) $|\pi(G)| > 2$. Then $G/N \neq H/N = QN/N \cdot R/N$, where $R/N \in \text{Syl}_p(G/N)$. By Statement 1) of Theorem $H/N \in sm\mathfrak{U}$. So Sylow q -subgroup QN/N and Sylow p -subgroup R/N of H/N are submodular in H/N . By Theorem 2.7, H/N is strongly supersoluble. Since by Theorem 2.6 $s\mathfrak{U}$ is a local formation and $H = QR$ is a Hall $\{p, q\}$ -subgroup of G , then apply Corollary 16.2.3 from [4]. From $H\Phi(G)/\Phi(G) \simeq H/H \cap \Phi(G) \simeq H/N/H \cap \Phi(G)/N \in s\mathfrak{U}$ it follows $H \in s\mathfrak{U}$. By Statement 1) of Theorem 2.5, $QN \in s\mathfrak{U}$. Then Q is submodular in QN , and so, Q is submodular in G . The arbitrariness of the choice of Q implies $G \in sm\mathfrak{U}$. This contradiction completes the proof of Statement 5).

Statement 6) follows from Statements 1)-3) and 5). Theorem is proved.

Lemma 3.2. *The following statements are held.*

- 1) *The class of groups $(G \in \mathfrak{S} \mid \text{Syl}(G) \subseteq \mathfrak{B})$ is a hereditary formation.*
- 2) *For any prime p the class of groups $(G \in \mathfrak{S} \mid \text{Syl}(G) \subseteq \mathfrak{A}(p-1) \cap \mathfrak{B})$ is a hereditary formation.*

Proof. Prove Statement 1). Denote $\mathfrak{H} = (G \in \mathfrak{S} \mid \text{Syl}(G) \subseteq \mathfrak{B})$. Clearly, if $H \leq G \in \mathfrak{H}$ and $N \trianglelefteq G$, then $H \in \mathfrak{H}$ and $G/N \in \mathfrak{H}$.

Let's show by induction on $|G|$, if $N_i \trianglelefteq G$ and $G/N_i \in \mathfrak{H}$, $i = 1, 2$, then $G/N_1 \cap N_2 \in \mathfrak{H}$. If $K = N_1 \cap N_2 \neq 1$, then from $|G/K| < |G|$ and $G/K/N_i/K \simeq G/N_i \in \mathfrak{H}$ it follows $G/K/N_1/K \cap N_2/K \simeq G/N_1 \cap N_2 \in \mathfrak{H}$. Let $N_1 \cap N_2 = 1$. Let $P \in \text{Syl}_p(G)$. From $G/N_i \in \mathfrak{H}$ it follows that $PN_i/N_i \simeq P/P \cap N_i$ is an elementary abelian group. Since the class of all abelian groups \mathfrak{A} is a formation, then $P/(P \cap N_1) \cap (P \cap N_2) \simeq P \in \mathfrak{A}$. We will show that P is an elementary abelian p -group. Let $z \in P$, $|z| = p^n$ and $Z = \langle z \rangle$. From $ZN_i/N_i \leq PN_i/N_i$ it follows that $|ZN_i/N_i| = |Z/Z \cap N_i| \leq p$. Since $N_1 \cap N_2 = 1$, then there exists $i \in \{1, 2\}$ such that $Z \cap N_i = 1$. Then $|Z| = p^n$, $n = 1$. Since P is a direct product of cyclic subgroups, we get $P \in \mathfrak{B}$. So $G \in \mathfrak{H}$.

Statement 2) is being proved similarly taking into account $\mathfrak{A}(p-1) \cap \mathfrak{B}$ is a hereditary formation. Lemma is proved.

Lemma 3.3. *A local formation $LF(f)$ with a local screen f such that $f(p) = (H \in \mathfrak{S} \mid \text{Syl}(H) \subseteq \mathfrak{A}(p-1) \cap \mathfrak{B})$ for any prime p , is a hereditary saturated formation.*

Proof. By Theorem 1.6, $LF(f)$ is a saturated formation.

Let's prove the heredity of $LF(f)$. Let $G \in LF(f)$ and $R \leq G$. Then G has a chief series $1 = G_0 < G_1 < \dots < G_{n-1} < G_n = G$ such that $G/C_G(G_i/G_{i-1}) \in f(p)$ for every $p \in \pi(G/G_{i-1})$ and $i = 1, \dots, n$. Let $R_{i-1} = R \cap G_{i-1}$, $i = 1, \dots, n+1$. Let $C_i = C_G(G_i/G_{i-1})$ and $C_i^* = C_R(R_i/R_{i-1})$, $i = 1, \dots, n$. It is easy to see that $R \cap C_i \leq C_i^*$. From $RC_i/C_i \leq G/C_i \in f(p)$ and the heredity of $f(p)$ it follows that $RC_i/C_i \simeq R/R \cap C_i \in f(p)$. Then $R/C_i^* \simeq R/R \cap C_i/C_i^*/R \cap C_i \in f(p)$. Hence $R/C_R(H/K) \in f(p)$ for every chief factor H/K of R and $p \in \pi(H/K)$. So $R \in LF(f)$. Lemma is proved.

Theorem 3.4. *Every minimal non $sm\mathfrak{U}$ -group is biprimary minimal non $s\mathfrak{U}$ -group.*

Proof. Let $G \in \mathcal{M}(sm\mathfrak{U})$ and q be the smallest prime divisor of $|G|$. Every subgroup H of a group G belongs $sm\mathfrak{U}$. By Corollary 2.1.1, H is Ore disperive. So H is q -nilpotent. Let's consider two cases.

1) G is q -nilpotent. Then $G = QP$, where $Q \in \text{Syl}_q(G)$, $P \trianglelefteq G$ and P is a Hall q' -subgroup of G . From $P \in sm\mathfrak{U}$ it follows the solubility of P . Then from $G/P \simeq Q$ we get the solubility of G .

Suppose that $\Phi(G) = 1$. Let N be a minimal normal subgroup of G . Then $|N| = p^n$ for some $p \in \pi(G)$. Let $G = NM$, where M is a maximal subgroup of G . In view of $G \in \mathcal{M}(sm\mathfrak{U})$ and $G/N \simeq M/M \cap N \in sm\mathfrak{U}$, N is the only minimal normal subgroup of G . Let R is an arbitrary Sylow r -subgroup of G . From $G/N \in sm\mathfrak{U}$ we conclude that RN/N is submodular in G/N . By 3) of Lemma 1.1, RN is submodular in G . If $RN \neq G$, then from $RN \in sm\mathfrak{U}$ we get R is submodular in G . This contradicts with $G \notin sm\mathfrak{U}$. Hence $RN = G$ is a biprimary group. Since every subgroup T of G belongs $sm\mathfrak{U}$, then, by Theorem 2.7, $T \in s\mathfrak{U}$. From $s\mathfrak{U} \subseteq sm\mathfrak{U}$ it follows that $G \notin s\mathfrak{U}$, i.e. $G \in \mathcal{M}(s\mathfrak{U})$.

Let $\Phi(G) \neq 1$. Then $\Phi(G/\Phi(G)) = 1$. Since $sm\mathfrak{U}$ is saturated, it follows $G/\Phi(G) \notin sm\mathfrak{U}$. Then $G/\Phi(G) \in \mathcal{M}(sm\mathfrak{U})$. As proved above, $G/\Phi(G)$ is a biprimary group and $G/\Phi(G) \notin s\mathfrak{U}$. Hence $G \in \mathcal{M}(s\mathfrak{U})$.

2) G is not q -nilpotent. By Theorem 5.4 of [10, гл. IV], G is a Schmidt group. Since every subgroup T of G is nilpotent, then $T \in s\mathfrak{U}$. Then $G \in \mathcal{M}(s\mathfrak{U})$. Theorem is proved.

Recall that a subgroup H of a group G is called $K\mathbb{P}$ -subnormal [11] in G , if there exists a chain of subgroups

$$H = H_0 \leq H_1 \leq \cdots \leq H_{n-1} \leq H_n = G \quad (0.1)$$

such that either H_{i-1} is normal in H_i , or $|H_i : H_{i-1}|$ is a prime for every $i = 1, \dots, n$.

If $H = G$ or in a chain (0.1) the index $|H_i : H_{i-1}|$ is a prime for every $i = 1, \dots, n$, then H is called \mathbb{P} -subnormal in G [12].

Lemma 3.5. *Let H be a submodular Sylow subgroup of a group G . Then the following conditions are held:*

- 1) H is $K\mathbb{P}$ -subnormal in G ;
- 2) if G is soluble, then H is \mathbb{P} -subnormal in G .

Proof. Prove 1) by induction on $|G|$. We can suppose that $H \neq G$. Then H is contained in a maximal modular subgroup M of G . By 1) of Lemma 1.1 and $|M| < |G|$, it follows that H is $K\mathbb{P}$ -subnormal in M . By Lemma 1.2 either M is normal in G , or G/M_G is non-abelian of order pq , where p and q are primes. Hence, if $M_G \neq M$ we have $|G : M| = |G/M_G : M/M_G|$ is a prime. This means that M is $K\mathbb{P}$ -subnormal in G . So H is $K\mathbb{P}$ -subnormal in G .

Statement 2) follows from 1), since in a soluble group $K\mathbb{P}$ -subnormal subgroup is \mathbb{P} -subnormal. Lemma is proved.

By Lemma 3.5, it follows that $sm\mathfrak{U} \subseteq w\mathfrak{U}$, where $w\mathfrak{U}$ is the class of all groups with \mathbb{P} -subnormal Sylow subgroups. Example 2.8 shows that $sm\mathfrak{U} \neq w\mathfrak{U}$.

Theorem 3.6. *The class of all groups with submodular Sylow subgroups is a local formation and has a local screen f such that $f(p) = (G \in \mathfrak{S} \mid \text{Syl}(G) \subseteq \mathfrak{A}(p-1) \cap \mathfrak{B})$ for every prime p .*

Proof. Since $f(p) = (G \in \mathfrak{S} \mid \text{Syl}(G) \subseteq \mathfrak{A}(p-1) \cap \mathfrak{B})$ is a formation, f is a local screen. Let a local formation $LF(f)$ be defined by a screen f . Denote $\mathfrak{F} = LF(f)$. By Theorem 2.10 [12], the class of groups $w\mathfrak{U}$ is a local formation and has a local screen h such that $h(p) = (G \in \mathfrak{S} \mid \text{Syl}(G) \subseteq \mathfrak{A}(p-1))$ for every prime p . Hence $\mathfrak{F} \subseteq w\mathfrak{U}$. In view of Proposition 2.8 [12], \mathfrak{F} consists of Ore dispersive groups.

Show that $\mathfrak{F} \subseteq sm\mathfrak{U}$. Let G be a group of the smallest order from $\mathfrak{F} \setminus sm\mathfrak{U}$. By Lemma 3.3, \mathfrak{F} is a hereditary formation. Hence G is a minimal non $sm\mathfrak{U}$ -group. Since $\mathfrak{F} \subseteq \mathfrak{S}$ and $sm\mathfrak{U}$ is a hereditary formation by 6) of Theorem 3.1, then G has the unique minimal normal subgroup N , $N = C_G(N)$ is an elementary abelian p -subgroup for some prime p , $\Phi(G) = 1$. Then $G = NM$, where M is a maximal subgroup of G . By Lemma 1.4, $O_p(M) = 1$. Since G is Ore dispersive, it follows that $N \in \text{Syl}_p(G)$ and p is the largest prime divisor of $|G|$. By Theorem 3.4, G is a biprimary minimal non $s\mathfrak{U}$ -group. Hence and from $G/C_G(N) \simeq M \in f(p)$ we get that M is an elementary abelian q -group and q divides $(p-1)$. Since M is \mathbb{P} -subnormal in G it follows that $|G : M| = p$ and $|N| = p$. Then $M \simeq G/N$ is isomorphically embedded in a cyclic group of order $p-1$. So $|M| = q$. Hence $G \in s\mathfrak{U} \subseteq sm\mathfrak{U}$. This contradicts the choice of G . So $\mathfrak{F} \subseteq sm\mathfrak{U}$.

Prove that $sm\mathfrak{U} \subseteq \mathfrak{F}$. Let G be a group of the smallest order from $sm\mathfrak{U} \setminus \mathfrak{F}$. Since $G \in sm\mathfrak{U}$, then G is soluble by Corollary 2.1.1. Since $sm\mathfrak{U}$ and \mathfrak{F} are saturated formations, then $\Phi(G) = 1$. In G there exists the unique minimal normal subgroup $N = C_G(N) = F(G)$, $|N| = p^n$ for some $p \in \pi(G)$. Then $G = NM$, where M is a maximal subgroup in G , $N \cap M = 1$. By Corollary 2.1.1, G is Ore dispersive. Then p is the largest prime divisor of $|G|$. In view of Lemma 1.4, $N \in \text{Syl}_p(G)$ and M is a p' -groups. Let $Q \in \text{Syl}_q(M)$. Then $Q \in \text{Syl}_q(G)$.

Suppose that $QN \neq G$. In view of $QN \in sm\mathfrak{U}$ and by Theorem 2.7, QN is strongly supersoluble. From $N \leq F(QN)$, $N = C_G(N)$ and $O_{p'}(QN) = 1$, it follows that $N = F_p(QN)$. By Lemma 1.7, $Q \simeq QN/F_p(QN) \in \mathfrak{A}(p-1) \cap \mathfrak{B}$. Then $M \in f(p)$ in view of the arbitrariness of the choice of Q . By Lemma 1.7 $G \in \mathfrak{F}$. This contradicts the choice of G .

Hence $QN = G$. By Theorem 2.7, G is strongly supersoluble. By Lemma 2.6, $G/C_G(N) = G/N \in \mathfrak{A}(p-1) \cap \mathfrak{B}$. So $G \in \mathfrak{F}$. This contradicts the choice of G . Theorem is proved.

Theorem 3.7. *Let G be a group in which every Sylow subgroup is submodular in G . Then the following conditions are held:*

- 1) every metanilpotent subgroup of G is strongly supersoluble;
- 2) every biprimary subgroup of G is strongly supersoluble;
- 3) if N is the smallest normal subgroups of G such that G/N is a group with elementary abelian Sylow subgroups, then N is nilpotent.

Proof. Statement 1) follows from the heredity of the class of groups $sm\mathfrak{U}$ and Theorem 3.6.

Statement 2) follows from the heredity of the class of groups $sm\mathfrak{U}$ and Theorem 2.9.

Prove Statement 3). By Lemma 3.2, the class of groups $\mathfrak{H} = (G \mid \text{Syl}(G) \subseteq \mathfrak{B})$ is a hereditary formation. By Statement 10 of [4, c. 36], \mathfrak{NH} has a local screen f such that $f(p) = \mathfrak{H}$ for every prime p . In view of Theorem 3.6, $sm\mathfrak{U} \subseteq \mathfrak{NH}$. Then $N = G^{\mathfrak{H}} \in \mathfrak{N}$. Theorem is proved.

Theorem 3.8. *Every Sylow subgroup of the group G is submodular in G if and only if the group G is Ore dispersive and every its biprimary subgroup is strongly supersoluble.*

Proof. Let the class of groups $\mathfrak{F} = (G \mid \text{the group } G \text{ is Ore dispersive and every biprimary subgroup of } G \text{ is strongly supersoluble})$.

If $G \in sm\mathfrak{U}$, then $G \in \mathfrak{F}$ in view of Corollary 2.1.1 and by 2) of Theorem 3.7.

Let G be a group of the smallest order belonging $\mathfrak{F} \setminus sm\mathfrak{U}$. Since $s\mathfrak{U} \subseteq sm\mathfrak{U}$, then $|\pi(G)| > 2$. Since G is Ore dispersive, then for the largest prime $p \in \pi(G)$ and $P \in$

$\in \text{Syl}_p(G)$, P is normal in G . Since G is soluble, there exists a Hall p' -subgroup H of G . From $H \neq G$ and $H \in \mathfrak{F}$ it follows that $H \in sm\mathfrak{U}$. Note that $|\pi(H)| \geq 2$. Let $Q \in \text{Syl}_q(G)$, $q \neq p$. Since $QP/P \in \text{Syl}_q(G/P)$ and $G/P \simeq H \in sm\mathfrak{U}$, then QP/P is submodular in G/P . By 3) of Lemma 1.1, QP is submodular in G . From $QP \in s\mathfrak{U}$, it follows that Q is submodular in QP . So Q is submodular in G . This means that $G \in sm\mathfrak{U}$. We got a contradiction with the choice of G . Theorem is proved.

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